

SPHERICAL AFFINE CONES IN EXCEPTIONAL CASES AND RELATED BRANCHING RULES

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ABSTRACT. Given a complex simply connected simple algebraic group G of exceptional type and a maximal parabolic subgroup $P \subset G$, we classify all triples (G, P, H) such that $H \subset G$ is a maximal reductive subgroup acting spherically on G/P . In addition we derive branching rules for $\text{res}_H^G(V_{k\omega_i}^*)$, $k \in \mathbb{N}$, where ω_i is the fundamental weight associated to P .

This is the first of two parts of a project to classify all such triples and corresponding branching rules for all simply connected simple algebraic groups.

CONTENTS

1. Introduction	1
2. Notation	2
3. Main results and outline of proof	2
4. The maximal reductive subgroups of the exceptional groups	6
5. The exceptional group of type G_2	7
6. The exceptional group of type F_4	9
7. The exceptional group of type E_6	12
8. The exceptional group of type E_7	19
9. The exceptional group of type E_8	23
References	24

1. INTRODUCTION

Given a reductive algebraic group G , a reductive subgroup H and some irreducible G -module V , then V is also a H -module in a natural way. An obvious problem is to find branching rules that describe the decomposition of the H -module V into irreducible components.

We will deal with this problem in the situation where G is a complex simply connected simple algebraic group of exceptional type. The subgroup structure of these groups has been studied in great detail and we want to consider maximal reductive subgroups of G . The maximal closed connected subgroups are listed in Theorem 1 of [Sei91]. These groups are either semisimple or parabolic. So the maximal reductive subgroups are easily obtained by adding the Levi factors of the maximal parabolic groups which are maximal reductive in G to the list of maximal semisimple subgroups. The modules V that we consider are those having as highest weights a multiple of a fundamental weight.

We will approach this problem by working with spherical varieties. We consider the flag variety G/P where P is a maximal parabolic subgroup of G . Of special interest to us are the flag varieties of that form, that are H -spherical, i.e. they contain an open orbit for a Borel subgroup of H . The property of being spherical can also be described in a representation-theoretic way. Namely a normal affine G -variety is spherical if and only if its coordinate ring is a multiplicity-free G -module [VK78]. Let \hat{Y} denote the affine cone over G/P . Then the flag variety is H -spherical if and only if all restrictions of the homogeneous components of the coordinate ring of \hat{Y} to H are multiplicity-free. These homogeneous components are exactly the irreducible submodules of the coordinate ring $\mathbb{C}[\hat{Y}]$ and they are of shape $V_{k\omega_i}^*$. In the case of sphericity we can derive branching rules for these modules.

So the content of this paper is twofold. We classify the spherical H -varieties G/P and furthermore we derive branching rules for the simple G -submodules of the coordinate ring of the affine cones in the spherical cases. The results are summarized in Table 1. A flag variety G/P is H -spherical if and only if the branching rules for the corresponding modules V are given in the table.

2. NOTATION

We work over the field of complex numbers throughout the article. G always denotes a simply connected simple algebraic group of exceptional type. Within G we choose a Borel subgroup B , a maximal torus T and thereby define a set $\{\alpha_1, \dots, \alpha_r\}$ of simple roots which are labeled according to Bourbaki-notation. The system of roots of G is denoted by Φ , the system of positive roots of G is denoted by Φ^+ and (a_1, \dots, a_r) stands for the root $\sum_{i=1}^r a_i \alpha_i$. Further X_α denotes a non-trivial element of the root space associated to α . Let Λ^+ be the set of dominant weights related to B and T . The irreducible G -module of highest weight $\lambda \in \Lambda^+$ is denoted by V_λ . The fundamental weights of G are $\omega_1, \dots, \omega_r$ and $\omega_1^*, \dots, \omega_r^*$ are the fundamental weights such that $(V_{\omega_i})^* = V_{\omega_i^*}$, where $(V_{\omega_i})^*$ is the dual of V_{ω_i} . If we write $k\omega_i$, then $k \in \mathbb{N}$.

Let H denote a reductive subgroup of G with root system Φ_H and analogous to G we use the notation $(b_1, \dots, b_s)_H := \sum_{i=1}^s b_i \beta_i$ where $\{\beta_1, \dots, \beta_s\}$ is a set of simple roots of Φ_H given by the Borel subgroup $B_H = B \cap H$. The fundamental weights of H are denoted by $\lambda_1, \dots, \lambda_s$, if H is semisimple. When H is a Levi subgroup, $\lambda_1, \dots, \lambda_s$ denote the fundamental weights of the semisimple part of H .

Lastly \mathfrak{b} denotes the Lie algebra of B_L , \mathfrak{u} the Lie algebra of U_L the unipotent radical of B_L and \mathfrak{h} the Lie algebra of the maximal torus T of B_L .

3. MAIN RESULTS AND OUTLINE OF PROOF

We will now summarize the results and give an outline of the proof. In this paper we will derive the branching rules stated in the following table. Further we show that if $\text{res}_H^G(V_{k\omega_i})$ is given in the table, then $G/P_{\omega_i}^*$ is a spherical H -variety. Conversely, if a maximal reductive subgroup $H \subset G$ does not appear in the table, then the varieties G/P_{ω_i} are not H -spherical.

Note that for the subgroups $D_5 \times \mathbb{C}^* \subset E_6$ and $E_6 \times \mathbb{C}^* \subset E_7$ the weight of the \mathbb{C}^* -action depends on the embedding of \mathbb{C}^* . The embedding that we chose is given in the corresponding sections.

TABLE 1

G	H	ω	$\text{res}_H^G(V_\omega)$
G_2	A_2	$k\omega_1$	$\bigoplus_{a_1+a_2 \leq k} V_{a_1\lambda_1+a_2\lambda_2}$
		$k\omega_2$	$\bigoplus_{a_1+a_2+a_3=k} V_{(a_1+a_3)\lambda_1+(a_2+a_3)\lambda_2}$
F_4	B_4	$k\omega_1$	$\bigoplus_{a_1+a_2=k} V_{a_1\lambda_2+a_2\lambda_4}$
		$k\omega_2$	$\bigoplus_{a_1+\dots+a_5=k} V_{(a_1+a_2)\lambda_1+(a_3+a_4)\lambda_2+(a_1+a_5)\lambda_3+(a_2+a_4)\lambda_4}$
		$k\omega_3$	$\bigoplus_{a_1+\dots+a_5=k} V_{(a_1+a_5)\lambda_1+a_2\lambda_2+a_3\lambda_3+(a_4+a_5)\lambda_4}$
		$k\omega_4$	$\bigoplus_{a_1+a_2 \leq k} V_{a_1\lambda_2+a_2\lambda_4}$
E_6	$A_5 \times A_1$	$k\omega_1$	$\bigoplus_{a_1+2a_2+a_3=k} V_{a_1\lambda_2+a_2\lambda_4+a_3\lambda_5} \otimes V_{a_3\lambda_6}$
		$k\omega_6$	$\bigoplus_{a_1+2a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_2+a_3\lambda_4} \otimes V_{a_1\lambda_6}$
	F_4	$k\omega_1$	$\bigoplus_{a_1 \leq k} V_{a_1\lambda_4}$
		$k\omega_2$	$\bigoplus_{a_1+a_2=k} V_{a_1\lambda_1+a_2\lambda_4}$
		$k\omega_3$	$\bigoplus_{a_1+a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_3+a_3\lambda_4}$
		$k\omega_5$	$\bigoplus_{a_1+a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_3+a_3\lambda_4}$
		$k\omega_6$	$\bigoplus_{a_1 \leq k} V_{a_1\lambda_4}$
	C_4	$k\omega_1$	$\bigoplus_{a_1+2a_2+2a_3=k} V_{a_1\lambda_2+a_2\lambda_4}$
		$k\omega_6$	$\bigoplus_{a_1+2a_2+2a_3=k} V_{a_1\lambda_2+a_2\lambda_4}$
	$D_5 \times \mathbb{C}^*$	$k\omega_1$	$\bigoplus_{a_1+a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_4} \otimes V_{-2a_1+a_2+4a_3}$
		$k\omega_2$	$\bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\lambda_2+a_2\lambda_4+a_3\lambda_5} \otimes V_{-3a_2+3a_3}$
		$k\omega_3$	$\bigoplus_{a_1+\dots+a_6=k} V_{(a_1+a_6)\lambda_1+a_2\lambda_2+a_3\lambda_3+(a_4+a_6)\lambda_4+a_5\lambda_5} \otimes V_{2a_1-4a_2+2a_3+5a_4-a_5-3a_6}$
		$k\omega_5$	$\bigoplus_{a_1+\dots+a_6=k} V_{(a_1+a_6)\lambda_1+a_2\lambda_2+a_3\lambda_3+a_4\lambda_4+(a_5+a_6)\lambda_5} \otimes V_{-2a_1+4a_2-2a_3+a_4-5a_5+a_6}$
		$k\omega_6$	$\bigoplus_{a_1+a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_4} \otimes V_{2a_1-a_2-4a_3}$
E_7	A_7	$k\omega_7$	$\bigoplus_{\substack{2a_1+a_2+\\2a_3+a_4=k}} V_{a_2\lambda_2+a_3\lambda_4+a_4\lambda_6}$
	$E_6 \times \mathbb{C}^*$	$k\omega_1$	$\bigoplus_{a_1+a_2+a_3 \leq k} V_{a_1\lambda_1+a_2\lambda_2+a_3\lambda_6} \otimes V_{2a_1-2a_3}$
		$k\omega_2$	$\bigoplus_{\substack{a_1+a_2+a_3+2a_4+\\a_5+a_6+a_7=k}} V_{a_1\lambda_1+(a_2+a_7)\lambda_2+a_3\lambda_3+a_4\lambda_4+a_5\lambda_5+a_6\lambda_6} \otimes V_{-a_1+3a_2+a_3-a_5-2a_6}$

TABLE 1

G	H	ω	$\text{res}_H^G(V_\omega)$
		$k\omega_7$	$\bigoplus_{\substack{a_1+a_2+\\a_3+a_4=k}} V_{a_1\lambda_1+a_2\lambda_6} \otimes V_{-a_1+a_2+3a_3-3a_4}$
	$D_6 \times A_1$	$k\omega_7$	$\bigoplus_{a_1+2a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_2+a_3\lambda_6} \otimes V_{a_1\lambda_7}$

To obtain the previous table we shall adapt the proof of Proposition 4.4 in [FL10] by Feigin and Littelmann. But first we will introduce some additional notation.

Let $P_i \supset B$ denote the maximal parabolic subgroup of G associated to the fundamental weight ω_i . We shall consider the natural action of H on the projective varieties $Y = G/P_i$. The affine cone over Y is denoted by \hat{Y} and the stabilizer of $\bar{1} \in G/P_i$ is denoted by $H_{\bar{1}}$. The group $H_{\bar{1}}$ is a parabolic subgroup of H . Its opposite parabolic subgroup in H is denoted by Q . Furthermore let Q^u be its unipotent radical and let L be the Levi-subgroup $H_{\bar{1}} \cap Q$ with Borel subgroup B_L defined by the simple roots of H that appear in L . If we consider the orbit $O = H \cdot \bar{1} \simeq H/H_{\bar{1}}$ with normal bundle \mathcal{N} having fiber N at $\bar{1}$ then N has the structure of an L -module since $L \subset H_{\bar{1}}$.

If no confusion can arise we will write P instead of P_i from now on.

The proof is divided into two parts. First we will determine in which cases Y is a spherical H -variety. This part of the proof is conducted in four steps.

Step 1: We apply the Brion-Luna-Vust Local Structure Theorem [BLV86] to get the following proposition.

Proposition 1: *There exists a locally closed affine subvariety $Z \subset Y$ such that $\bar{1} \in Z$, Z is stable under the action of L , $Q^u \cdot Z$ is open in Y and the canonical map $Q^u \times Z \rightarrow Q^u \cdot Z$ is an isomorphism of varieties.*

Proof: Note that since the Borel subgroup B_H is a subgroup of P , it is contained in the stabilizer $H_{\bar{1}}$ of $\bar{1} \in Y$. Thus $H_{\bar{1}}$ is a parabolic subgroup of H .

Now we can apply the Local Structure Theorem to this situation and obtain the proposition. \square

Step 2: We have the following proposition.

Proposition 2: *The variety Y is H -spherical if and only if Z is a spherical L -variety.*

Proof: Assume Z is spherical, i.e. a Borel subgroup of L has a dense orbit in Z . Let B_L be the Borel subgroup $B_H \cap L \subset L$ and let B_L^- be the opposite Borel subgroup. Then $B_H^- = Q^u B_L^-$ is a Borel subgroup of H . Let $z \in Z$ be an element such that $B_L^- \cdot z$ is dense in Z . Since $Q^u \cdot Z$ is dense in Y , so is $B_H^- \cdot z = Q^u (B_L^- \cdot z)$. Hence Y is a spherical H -variety.

If on the other hand Y is H -spherical, then $B_H^- \cdot y = Q^u (B_L^- \cdot y)$ is open in Y for some $y \in Y$. Since $Q^u \cdot Z$ is open in Y we can assume that $y \in Z$. Now if $Q^u (B_L^- \cdot y)$ is dense in Y it follows that $B_L^- \cdot y$ is dense in Z . \square

Step 3: Now N is isomorphic to the tangent space $T_{\overline{1}}Z$ and thanks to Luna's Slice Theorem Y is H -spherical if and only if N is L -spherical.

Step 4: It remains to compute N and to check in which cases it is a spherical L -module. Note that we have

$$N \simeq (\mathrm{Lie} G / \mathrm{Lie} P_i) / (\mathrm{Lie} H / \mathrm{Lie} H_{\overline{1}}).$$

So if $\Phi_H \subset \Phi$, then we can describe N as the root spaces that occur in $T_{\overline{1}}Y = \mathrm{Lie} G / \mathrm{Lie} P_i$ but not in $T_{\overline{1}}(H / H_{\overline{1}})$. These are all the root spaces $\mathbb{C}X_\alpha$ such that α is negative and $\mathbb{C}X_\alpha \not\subset \mathrm{Lie} P_i$ as well as $\mathbb{C}X_\alpha \not\subset \mathrm{Lie} H$.

Remark. There is an algorithm by F. Knop [Kno97, Thm. 3.3] to check whether a given L -module is spherical. But in order for this paper to be self-contained we compute an explicit $X \in N$ such that $B_L.X$ is a dense orbit in N in the spherical cases.

The second part is to compute the restrictions of the G -modules $V_{k\omega_i^*}$ to H . It is well-known that

$$\mathbb{C}[\widehat{Y}] = \bigoplus_{k \geq 0} V_{k\omega_i^*}$$

where $V_{k\omega_i^*}$ corresponds to the homogeneous functions of degree k on \widehat{Y} . In order to derive branching rules for $V_{k\omega_i^*}$ we need to determine the U_H -invariants of $V_{k\omega_i^*}$.

Because \widehat{Y} is a spherical $(H \times \mathbb{C}^*)$ -variety and because $U_H = U_{H \times \mathbb{C}^*}$, we know from Lemma 1 in [Lit94] that the ring $\mathbb{C}[\widehat{Y}]^{U_H}$ is a polynomial ring with some set of generators f_j of degree d_j , $1 \leq j \leq s$, where s is the number of generators. Thus we have the following branching rules in this situation.

Theorem 3: *Let η_j denote the weight of f_j with respect to H and suppose G/P_i is a spherical H -variety. Then we get*

$$\mathrm{res}_H^G(V_{k\omega_i^*}) = \bigoplus_{a_1 d_1 + \dots + a_s d_s = k} V_{a_1 \eta_1 + \dots + a_s \eta_s}.$$

We need to compute the number of generators, i.e. the dimension of $\mathbb{C}[\widehat{Y}]^{U_H}$.

Proposition 4: *We have*

$$\dim \mathbb{C}[\widehat{Y}]^{U_H} = \dim N - \dim(\text{generic } U_L\text{-orbit}) + 1.$$

Proof: We know that $\dim \mathbb{C}[\widehat{Y}]^{U_H} = \mathrm{trdeg} \mathbb{C}(\widehat{Y})^{U_H}$ and by a theorem of Rosenthal we know that $\mathrm{trdeg} \mathbb{C}(\widehat{Y})^{U_H} = \dim \widehat{Y} - \dim(\text{generic } U_H\text{-orbit})$ (paragraph II.4.3.E in [Kra84, p. 143]).

So the proposition is an immediate corollary of the following lemma. \square

Lemma 5: *Let Y , N , U_L and U_H be defined as above. Let O_1 be a generic U_H -orbit in Y and O_2 be a generic U_L -orbit in N . Then*

$$\dim Y - \dim O_1 = \dim N - \dim O_2.$$

Proof: Let $O \subset Y$ be the open subset of X such that $\dim U_H.x$ is maximal for all $x \in O$ (i.e. $U_H.x$ is an generic orbit). We have $O \cap Q^u.Z \neq \emptyset$, because $Q^u.Z$ is open and dense in Y .

Let $x = qz$ be an element in $O \cap Q^u.Z$. We know that $U_H = U_L.Q^u = Q^u.U_L$. So we have $U_H.x = U_H.(qz) = U_L.Q^u(qz) = U_L.Q^u.z = U_H.z$ and we can assume that $U_H.x$ is a generic U_H -orbit in Y with $x \in Z$.

Suppose y is an element of the stabilizer $(U_H)_x$ of x . Then we have $y = q.u$ for some $q \in Q^u$, $u \in U_L$. So it follows from the Local Structure Theorem that $q = \text{id}$ and $ux = x$. Thus we get $(U_H)_x = (U_L)_x$.

With $\dim Y = \dim Z + \dim Q^u$ (Local Structure Theorem) we get

$$\begin{aligned}
\dim Y - \dim U_H.x &= \dim Q^u + \dim Z - \dim U_H.z \\
&= \dim Z - (\dim U_H.x - \dim Q^u) \\
&= \dim Z - (\dim U_H - \dim(U_H)_x - \dim Q^u) \\
&= \dim Z - (\dim U_H - \dim Q^u - \dim(U_L)_x) \\
&= \dim Z - (\dim U_L - \dim(U_L)_x) \\
&= \dim Z - \dim U_L.x.
\end{aligned}$$

□

4. THE MAXIMAL REDUCTIVE SUBGROUPS OF THE EXCEPTIONAL GROUPS

We want to list all maximal reductive subgroups of the exceptional algebraic groups. G. Seitz listed all maximal closed connected subgroups in arbitrary characteristics. We recall his results for the case that the ground field is \mathbb{C} ([Sei91], Thm. 1).

Theorem 6: *Let G be a simple algebraic group of exceptional type and let X be maximal among the proper closed connected subgroups of G . Then either X contains a maximal torus of G or X is semisimple and the pair (G, X) is given below. Moreover, maximal subgroups of each type exist and are unique up to conjugacy in $\text{Aut}(G)$.*

G	X simple	X not simple
G_2	A_1	
F_4	A_1	$A_1 \times G_2$
E_6	A_2, G_2, F_4, C_4	$A_2 \times G_2$
E_7	A_1, A_2	$A_1 \times A_1, A_1 \times G_2, A_1 \times F_4, G_2 \times C_3$
E_8	A_1, B_2	$A_1 \times A_2, G_2 \times F_4$

Since the maximal subgroups that do not contain a maximal torus are semisimple they are also maximal reductive subgroups of G .

It remains to identify the maximal reductive subgroups that are contained in a maximal subgroup of maximal rank. These groups fall in two categories. Some are the maximal parabolic subgroups of G and the others are so called subsystem subgroups. There is an algorithm (cf. paragraph no. 17 of [Dyn57] or [BdS49]) that determines these subgroups: Start with the Dynkin diagram of G and adjoin the smallest root δ to obtain the extended Dynkin diagram. By removing a node from the extended diagram you arrive at the Dynkin diagram of a subgroup of G . By Theorem 5.5 and the subsequent remark in

[Dyn57] these groups are maximal. Since they are semisimple they are also maximal reductive.

To complete the list we need to consider the maximal parabolic subgroups of G . Any reductive subgroup of a parabolic can be assumed to be a subgroup of its Levi factor by Theorem 1 in [LS96]. By considering the Dynkin diagrams it is transparent that the Levi subgroups need not be maximal reductive but can be subgroups of a subsystem subgroup. A simple case by case check shows that there are only two Levi groups, that are maximal reductive.

Summarizing this we have the following maximal reductive subgroups containing a maximal torus.

G	subsystem subgroups	Levi subgroups
G_2	$A_2, A_1 \times A_1$	
F_4	$A_1 \times C_3, A_2 \times A_2, A_3 \times A_1, B_4$	
E_6	$A_5 \times A_1, A_2 \times A_2 \times A_2$	$D_5 \times \mathbb{C}^*$
E_7	$D_6 \times A_1, A_5 \times A_2, A_3 \times A_3 \times A_1, A_7$	$E_6 \times \mathbb{C}^*$
E_8	$A_1 \times E_7, A_2 \times E_6, A_3 \times D_5, A_4 \times A_4$ $A_5 \times A_2 \times A_1, A_7 \times A_1, D_8, A_8$	

5. THE EXCEPTIONAL GROUP OF TYPE G_2

We will now consider the simply connected simple algebraic group G of type G_2 . The long roots of its root system form a subsystem of type A_2 and we will consider the subsystem subgroup H obtained in this way. The simple roots of H are given by

$$(1, 0)_{A_2} = (3, 1) \text{ and } (0, 1)_{A_2} = (0, 1).$$

Using the same methods as before we can prove:

Theorem 7: *The varieties G/P_1 and G/P_2 are H -spherical.*

Proof:

Case G/P_1 : We compute

$$L = \langle T, U_{\pm(0,1)} \rangle.$$

and

$$N = \mathbb{C}X_{-(1,0)_{G_2}} \oplus \mathbb{C}X_{-(1,1)_{G_2}} \oplus \mathbb{C}X_{-(2,1)_{G_2}}.$$

If we define $X := X_{-(1,1)} + X_{-(2,1)}$ we have $[\mathfrak{b}, X] = N$, which shows that N is L -spherical. It follows that G/P_1 is a spherical H -variety.

Case G/P_2 : In this case we can compute that $L = T$ and

$$N = \mathbb{C}X_{-(1,1)} \oplus \mathbb{C}X_{-(2,1)}.$$

The module N consists of two linearly independent root spaces and since T is 2-dimensional N is obviously L -spherical. That implies that G/P_2 is a spherical H -variety. \square

Theorem 8: Let G be of type G_2 and H of type A_2 . Then we have the following branching rules:

$$\begin{aligned} i) \quad \text{res}_H^G(V_{k\omega_1}) &= \bigoplus_{a_1+a_2 \leq k} V_{a_1\lambda_1+a_2\lambda_2}, \\ ii) \quad \text{res}_H^G(V_{k\omega_2}) &= \bigoplus_{a_1+a_2+a_3=k} V_{(a_1+a_3)\lambda_1+(a_2+a_3)\lambda_2}. \end{aligned}$$

Remark. In G_2 the fundamental weights are self-dual.

Proof: i) We use “LiE” to compute the restriction of V_{ω_1} and get

$$\text{res}_H^G(V_{\omega_1}) = \mathbb{C} \oplus V_{\lambda_1} \oplus V_{\lambda_2}.$$

Let f_0, f_1, f_2 be highest weight vectors of these representations. We need to show that $\mathbb{C}[\hat{Y}]^{U_H}$ is generated by these elements, i.e. we need to show that the dimension of $\mathbb{C}[\hat{Y}]^{U_H}$ is 3.

By considering $X_{-(1,0)} \in N$ we immediately see that the U_L -orbit of this element is of codimension 2. Thus $\dim \mathbb{C}[\hat{Y}]^{U_H} = 3$ and since we have already found three algebraically independent elements the branching rules follow immediately.

ii) We use “LiE” to compute

$$\text{res}_H^G(V_{\omega_2}) = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_1+\lambda_2}.$$

Let f_1, f_2, f_3 be highest weight vectors of these modules. We know that U_L is the maximal torus in this case and so the unipotent radical is just the identity. A generic orbit in N is of dimension 0. And since N is 2-dimensional, its codimension is 2. That means a generic U_H -orbit has codimension 3 in \hat{Y} and that is also the dimension of $\mathbb{C}[\hat{Y}]^{U_H}$. We have already found three linearly independent elements which form a generating set. The branching rules follow immediately. \square

Proposition 9: The varieties G/P_i are not spherical H -varieties if H is any other maximal reductive subgroup of G_2 .

Proof: We have the following maximal reductive subgroups besides A_2 : $A_1 \times A_1$ and A_1 . If we compute the dimensions of a Borel subgroup in each case and the dimensions of G/P_i we obtain:

	G/P_1	G/P_2
dim	5	5

and

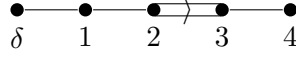
H	$A_1 \times A_1$	A_1
dim B_H	4	2

So $\dim B_H < \dim G/P_i$, $i = 1, 2$ for these subgroups. \square

6. THE EXCEPTIONAL GROUP OF TYPE F_4

In this section let G be the group of type F_4 .

Let H be the subgroup of type B_4 in G . This is a subsystem subgroup so from the Dynkin-diagram of F_4 we pass on to the extended Dynkin-diagram by adding the smallest root δ to the system of simple roots.



By removing the simple root α_4 we obtain a root-subsystem of type B_4 and thus we find the corresponding subgroup $H \subset G$.

Explicitly we can choose the roots

$$\begin{aligned} (1, 0, 0, 0)_{B_4} &= (0, 1, 2, 2), & (0, 1, 0, 0)_{B_4} &= (1, 0, 0, 0), \\ (0, 0, 1, 0)_{B_4} &= (0, 1, 0, 0), & (0, 0, 0, 1)_{B_4} &= (0, 0, 1, 0), \end{aligned}$$

which form a set of simple roots of a root subsystem of type B_4 in F_4 .

We have the following theorem:

Theorem 10: *The varieties G/P_i , $i = 1, \dots, 4$, are spherical H -varieties.*

Proof: We need to check that N is a spherical L -module in each case.

Case G/P_1 : In this case we have

$$\begin{aligned} L &= \langle T, U_{\pm(0,1,2,2)}, U_{\pm(0,1,0,0)}, U_{\pm(0,0,1,0)}, \\ &\quad U_{\pm(0,1,1,0)}, U_{\pm(0,1,2,0)} \rangle \end{aligned}$$

and

$$N = \mathbb{C}X_{-(1,2,3,1)} \oplus \mathbb{C}X_{-(1,2,2,1)} \oplus \mathbb{C}X_{-(1,1,2,1)} \oplus \mathbb{C}X_{-(1,1,1,1)}.$$

The Borel subgroup B_L of L obviously contains the maximal torus T of G . Since N consists of four root spaces with linearly independent roots and T is 4-dimensional we know that there is a dense B_L -orbit in N . Hence N is L -spherical and that implies that G/P_1 is H -spherical.

Case G/P_2 : Here we have

$$L = \langle T, U_{\pm(1,0,0,0)}, U_{\pm(0,0,1,0)} \rangle.$$

We compute N in the same way as in the previous case and get

$$\begin{aligned} N &= \mathbb{C}X_{-(0,1,1,1)} \oplus \mathbb{C}X_{-(0,1,2,1)} \oplus \mathbb{C}X_{-(1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,2,1)} \oplus \\ &\quad \mathbb{C}X_{-(1,2,2,1)} \oplus \mathbb{C}X_{-(1,2,3,1)}. \end{aligned}$$

We check the sphericity on the level of Lie algebras. Consider the element

$$X := X_{-(1,1,2,1)} + X_{-(0,1,2,1)} + X_{-(1,1,1,1)} + X_{-(1,2,3,1)}$$

in N . Then $[\mathfrak{b}, X] = N$. That means that N is a spherical L -variety and therefore G/P_2 is a spherical H -variety.

Case G/P_3 : We get

$$\begin{aligned} N &= \mathbb{C}X_{-(0,0,1,1)} \oplus \mathbb{C}X_{-(0,1,1,1)} \oplus \mathbb{C}X_{-(0,1,2,1)} \oplus \\ &\quad \mathbb{C}X_{-(1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,2,1)} \oplus \mathbb{C}X_{-(1,2,2,1)} \oplus \\ &\quad \mathbb{C}X_{-(1,2,3,1)}. \end{aligned}$$

If we consider

$$X := X_{-(1,2,3,1)} + X_{-(1,2,2,1)} + X_{-(1,1,1,1)} + X_{-(0,1,2,1)} \in N$$

we have that $[\mathfrak{b}, X] = N$, i.e. N is a spherical L -variety and that means that G/P_3 is a spherical H -variety.

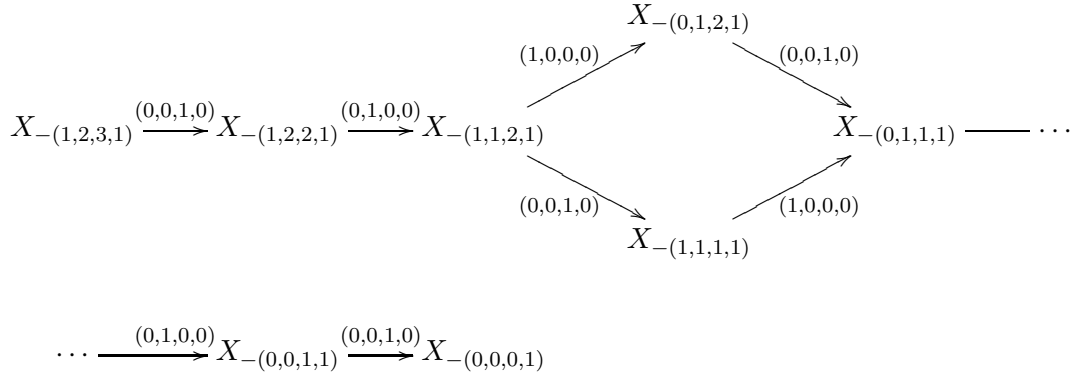
Case G/P_4 : In this case we have

$$\begin{aligned} L = \langle & T, U_{\pm(1,0,0,0)}, U_{\pm(0,1,0,0)}, U_{\pm(0,0,1,0)}, \\ & U_{\pm(1,1,0,0)}, U_{\pm(0,1,1,0)}, U_{\pm(1,1,1,0)}, \\ & U_{\pm(0,1,2,0)}, U_{\pm(1,1,2,0)}, U_{\pm(1,2,2,0)} \rangle \end{aligned}$$

and

$$\begin{aligned} N = & \mathbb{C}X_{-(0,0,0,1)} \oplus \mathbb{C}X_{-(0,0,1,1)} \oplus \mathbb{C}X_{-(0,1,1,1)} \oplus \\ & \mathbb{C}X_{-(0,1,2,1)} \oplus \mathbb{C}X_{-(1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,2,1)} \oplus \\ & \mathbb{C}X_{-(1,2,2,1)} \oplus \mathbb{C}X_{-(1,2,3,1)}. \end{aligned}$$

The module N has the following structure.



We have $L = \mathbb{C}^* \times SO_7$ and N is an irreducible L -module of dimension 8. There exists only one such module which is the Spin_7 -module. That N is a spherical L -module was proven by Victor Kac [Kac80, Thm. 3, p. 208]. It follows that G/P_4 is a spherical H -module. \square

The spherical cases imply the following branching rules.

Theorem 11: *Let G be of type F_4 and H of type B_4 . Then we have the following branching rules:*

$$\begin{aligned} i) \quad \text{res}_H^G(V_{k\omega_1}) &= \bigoplus_{a_1+a_2=k} V_{a_1\lambda_2+a_2\lambda_4}, \\ ii) \quad \text{res}_H^G(V_{k\omega_2}) &= \bigoplus_{a_1+\dots+a_5=k} V_{(a_1+a_2)\lambda_1+(a_3+a_4)\lambda_2+(a_1+a_5)\lambda_3+(a_2+a_4)\lambda_4}, \\ iii) \quad \text{res}_H^G(V_{k\omega_3}) &= \bigoplus_{a_1+\dots+a_5=k} V_{(a_1+a_5)\lambda_1+a_2\lambda_2+a_3\lambda_3+(a_4+a_5)\lambda_4}, \\ iv) \quad \text{res}_H^G(V_{k\omega_4}) &= \bigoplus_{a_1+a_2 \leq k} V_{a_1\lambda_2+a_2\lambda_4}. \end{aligned}$$

Remark. In F_4 the fundamental weights are self-dual.

Proof:

i): Standard computations yield

$$\text{res}_H^G(V_{\omega_1}) = V_{\lambda_2} \oplus V_{\lambda_4}.$$

Let now $f_1, f_2 \in V_{\omega_1}$ be highest weight vectors of V_{λ_2} and V_{λ_4} respectively. We will show that $\mathbb{C}[\widehat{Y}]^{U_H}$ is generated by these degree 1 elements. We know that $\mathbb{C}[\widehat{Y}]^{U_H}$ is a polynomial ring. The grading and weights of f_1 and f_2 imply that they are algebraically independent. To rule out the possibility that there are generators of degree two or higher we need to show that the Krull dimension of $\mathbb{C}[\widehat{Y}]^{U_H}$ is 2.

Thus we need to find a generic U_L -orbit in N and compute its codimension. Since we have found 2 algebraically independent elements in $\mathbb{C}[\widehat{Y}]^{U_H}$, we already know that the codimension must be at least 2.

Consider the Lie algebra \mathfrak{l} of L . From above we know that the Lie algebra \mathfrak{u} of U_L , is

$$\mathfrak{u} = \mathbb{C}X_{(0,1,2,2)} \oplus \mathbb{C}X_{(0,1,0,0)} \oplus \mathbb{C}X_{(0,0,1,0)} \oplus \mathbb{C}X_{(0,1,1,0)} \oplus \mathbb{C}X_{(0,1,2,0)}.$$

Define $X := X_{-(1,2,3,1)} \in N$. Then

$$\begin{aligned} [X_{(0,1,2,2)}, X] &= 0, & [X_{(0,1,0,0)}, X] &= 0, \\ [X_{(0,0,1,0)}, X] &= X_{-(1,2,2,1)}, & [X_{(0,1,1,0)}, X] &= X_{-(1,1,2,1)}, \\ [X_{(0,1,2,0)}, X] &= X_{-(1,1,1,1)}, \end{aligned}$$

which shows that the orbit of X is of dimension 3. Thus a generic orbit has dimension at least 3 with codimension at most 1. By Proposition 4 we know that in this case $\dim \mathbb{C}[\widehat{Y}]^{U_H} \leq 2$. But since we have found two generators the dimension is exactly 2 and the restriction rules follow.

ii): In this case we need to find generators of $\mathbb{C}[\widehat{Y}]^{U_H}$. One can use the software “LiE” to compute

$$\text{res}_H^G(V_{\omega_2}) = V_{\lambda_1+\lambda_3} \oplus V_{\lambda_1+\lambda_4} \oplus V_{\lambda_2} \oplus V_{\lambda_2+\lambda_4} \oplus V_{\lambda_3}.$$

Let f_1, \dots, f_5 be highest weight vectors of these irreducible modules.

Consider $X := X_{-(1,1,2,1)} + X_{-(1,2,3,1)} \in N$ and let \mathfrak{u} be the Lie-algebra of U_L the unipotent radical of L . The stabilizer of this element is just 0, which means that the dimension of a generic U_L -orbit is 2 with codimension 4. This implies that the codimension of a generic U_H -orbit in \widehat{Y} is 5. Thus $\mathbb{C}[\widehat{Y}]^{U_H}$ is generated by its degree 1 elements and the assertion follows.

iii): We need to find generators of $\mathbb{C}[\widehat{Y}]^{U_H}$. One can use “LiE” to compute

$$\text{res}_H^G(V_{\omega_3}) = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_4} \oplus V_{\lambda_1+\lambda_4}.$$

Let f_1, \dots, f_5 be highest weight vectors of these irreducible modules.

Consider $X := X_{-(1,1,1,1)} + X_{-(1,2,2,1)} \in N$ and take an element $u \in \mathfrak{u}$ with $u = aX_{(1,0,0,0)} + bX_{(0,1,0,0)} + cX_{(1,1,0,0)}$. Then

$$\begin{aligned} [u, X] &= 0 \\ \Rightarrow &= aX_{-(0,1,1,1)} + bX_{-(1,1,2,1)} + c(X_{-(0,1,2,1)} + X_{-(0,0,1,1)}) \\ \Rightarrow &a = b = c = 0 \Rightarrow u = 0 \end{aligned}$$

and hence a generic U_L -orbit has dimension 3 with codimension 4. That means that $\mathbb{C}[\widehat{Y}]^{U_H}$ is of dimension 5 and generated by the elements f_i .

iv): In this case we need to find generators of $\mathbb{C}[\hat{Y}]^{U_H}$. We use “LiE” to compute

$$\text{res}_H^G(V_{\omega_4}) = \mathbb{C} \oplus V_{\lambda_1} \oplus V_{\lambda_4}.$$

Let f_1, \dots, f_3 be highest weight vectors of these irreducible modules.

Consider $X := X_{-(1,2,3,1)}$. We know that for

$$X_{(1,0,0,0)}, X_{(0,1,0,0)}, X_{(1,1,0,0)} \in \mathfrak{u}$$

we have

$$[X_{(1,0,0,0)}, X] = [X_{(0,1,0,0)}, X] = [X_{(1,1,0,0)}, X] = 0$$

and

$$\begin{aligned} [X_{(0,0,1,0)}, X] &= X_{-(1,2,2,1)}, & [X_{(0,1,1,0)}, X] &= X_{-(1,1,2,1)}, \\ [X_{(0,1,2,0)}, X] &= X_{-(1,1,1,1)}, & [X_{(1,1,1,0)}, X] &= X_{-(0,1,2,1)}, \\ [X_{(1,1,2,0)}, X] &= X_{-(0,1,1,1)}, & [X_{(1,2,2,0)}, X] &= X_{-(0,0,1,1)} \end{aligned}$$

and thus the generic stabilizer is at most of dimension 3. The generic orbit is at least of dimension 6 and thus its codimension is at most 2. This means that a generic U_H -orbit in \hat{Y} is of dimension less or equal to 3.

Since we have found 3 algebraically independent elements the dimension of $\mathbb{C}[\hat{Y}]^{U_H}$ is exactly 3 and this finishes the proof. \square

Proposition 12: *The varieties G/P_i are not spherical H -varieties if H is any other maximal reductive subgroup of F_4 .*

Proof: We have the following maximal reductive subgroups besides B_4 : $A_1 \times C_3$, $A_2 \times A_2$, $A_3 \times A_1$, $A_1 \times G_2$ and A_1 . If we compute the dimensions of a Borel subgroup in each case and the dimensions of G/P_i we obtain:

	G/P_1	G/P_2	G/P_3	G/P_4	
dim	15	20	20	15	
H	$A_1 \times C_3$	$A_2 \times A_2$	$A_3 \times A_1$	$A_1 \times G_2$	A_1
dim B_H	14	10	11	10	2

So we have $\dim B_H < \dim G/P_i$ for $i = 1, \dots, 4$ in each case. \square

7. THE EXCEPTIONAL GROUP OF TYPE E_6

We will now turn to the group of type E_6 . First we calculate the dimensions of the Borel subgroups of the maximal reductive subgroups as well as the dimensions of G/P_i for $i = 1, \dots, 6$.

H	$A_5 \times A_1$	$A_2 \times A_2 \times A_2$	$D_5 \times \mathbb{C}^*$	$A_2 \times G_2$	G_2	A_2	F_4	C_4
dim B_H	22	15	26	13	8	5	28	20

and

	G/P_1	G/P_2	G/P_3	G/P_4	G/P_5	G/P_6
dim	16	21	25	29	25	16

Thus we get the following proposition.

Proposition 13: *Let G be the simply connected simple algebraic group of type E_6 and let H be a maximal reductive subgroup of type $A_2 \times A_2 \times A_2$, $A_2 \times G_2$, G_2 or A_2 .*

Then G/P_i is not H -spherical for $i = 1, \dots, 6$.

Proof: In these cases we have $\dim B_H < \dim G/P_i$ for $i = 1, \dots, 6$. \square

Now we will consider the remaining groups and first we start with the subsystem subgroup of type $A_5 \times A_1$.

Theorem 14: *Let G be the simply connected simple algebraic group of type E_6 and let H be the maximal reductive subgroup of type $A_5 \times A_1$. Then G/P_1 and G/P_6 are spherical H -varieties. The varieties $G/P_2, \dots, G/P_5$ are not H -spherical.*

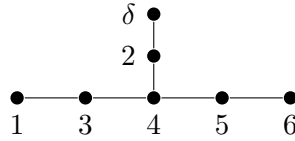
Proof: The dimension of a Borel subgroup of a group of type $A_5 \times A_1$ is 22. Since we have $\dim G/P_3 = 25$, $\dim G/P_4 = 29$, $\dim G/P_5 = 25$ these varieties cannot be spherical.

We know that $\omega_2^* = \omega_2$ in type E_6 . Now if G/P_2 was a spherical H -variety, $\text{res}_H^G(V_{k\omega_2})$ would be multiplicity-free for all $k \in \mathbb{N}$ by what has been said above. But with “LiE” we compute

$$\text{res}_H^G(V_{4\omega_2}) = \dots \oplus 2(V_{2\lambda_3} \otimes V_{3\lambda_6}) \oplus \dots$$

which means that there are multiplicities in this case.

To prove that G/P_1 and G/P_6 are spherical H -varieties we proceed as in the cases above. We will show how H is embedded in G . For doing so we consider the extended Dynkin-diagram of type E_6 again by adding the smallest root δ to the simple roots. Now omitting the root α_2 we obtain the embedding of $A_5 \times A_1$ in E_6 .



Explicitly we get the following set of simple roots:

$$\begin{aligned} (1, 0, 0, 0, 0, 0)_{A_5 \times A_1} &= (1, 0, 0, 0, 0, 0) & (0, 1, 0, 0, 0, 0)_{A_5 \times A_1} &= (0, 0, 1, 0, 0, 0) \\ (0, 0, 1, 0, 0, 0)_{A_5 \times A_1} &= (0, 0, 0, 1, 0, 0) & (0, 0, 0, 1, 0, 0)_{A_5 \times A_1} &= (0, 0, 0, 0, 1, 0) \\ (0, 0, 0, 0, 1, 0)_{A_5 \times A_1} &= (0, 0, 0, 0, 0, 1) & (0, 0, 0, 0, 0, 1)_{A_5 \times A_1} &= (1, 2, 2, 3, 2, 1) \end{aligned}$$

Case G/P_1 : We compute

$$\begin{aligned} L = \langle & T, U_{\pm(0,0,1,0,0,0)}, U_{\pm(0,0,0,1,0,0)}, U_{\pm(0,0,0,0,1,0)}, U_{\pm(0,0,0,0,0,1)}, \\ & U_{\pm(0,0,1,1,0,0)}, U_{\pm(0,0,0,1,1,0)}, U_{\pm(0,0,0,0,1,1)}, \\ & U_{\pm(0,0,1,1,1,0)}, U_{\pm(0,0,0,1,1,1)}, U_{\pm(0,0,1,1,1,1)} \rangle \end{aligned}$$

and

$$\begin{aligned} N = & \mathbb{C}X_{-(1,1,1,1,0,0)} \oplus \mathbb{C}X_{-(1,1,1,1,1,0)} \oplus \mathbb{C}X_{-(1,1,1,2,1,0)} \oplus \\ & \mathbb{C}X_{-(1,1,1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,2,2,1,0)} \oplus \mathbb{C}X_{-(1,1,1,2,1,1)} \oplus \\ & \mathbb{C}X_{-(1,1,2,2,1,1)} \oplus \mathbb{C}X_{-(1,1,1,2,2,1)} \oplus \mathbb{C}X_{-(1,1,2,2,2,1)} \oplus \\ & \mathbb{C}X_{-(1,1,2,3,2,1)}. \end{aligned}$$

Now let $X := X_{-(1,1,2,3,2,1)} + X_{-(1,1,1,1,1,1)}$. We have

$$[\mathfrak{h}, X] = \langle X_{-(1,1,2,3,2,1)}, X_{-(1,1,1,1,1,1)} \rangle,$$

since the roots are linearly independent. Next we compute

$$\begin{aligned} [X_{(0,0,0,1,0,0)}, X] &= X_{-(1,1,2,2,2,1)} & [X_{(0,0,0,1,1,0)}, X] &= X_{-(1,1,2,2,1,1)} \\ [X_{(0,0,1,1,0,0)}, X] &= X_{-(1,1,1,2,2,1)} & [X_{(0,0,1,1,1,0)}, X] &= X_{-(1,1,1,2,1,1)} \\ [X_{(0,0,0,1,1,1)}, X] &= X_{-(1,1,2,2,1,0)} & [X_{(0,0,1,1,1,1)}, X] &= X_{-(1,1,1,2,1,0)} \\ [X_{(0,0,0,0,0,1)}, X] &= X_{-(1,1,1,1,1,0)} & [X_{(0,0,0,0,1,1)}, X] &= X_{-(1,1,1,1,0,0)} \end{aligned}$$

and these computations show that we have ten linearly independent vectors in $[\mathfrak{b}, X] \Rightarrow [\mathfrak{b}, X] = N \Rightarrow N$ is a spherical L -module. Hence G/P_1 is a spherical H -variety.

Case G/P_6 : The H -sphericity of G/P_6 is an immediate corollary of the following theorem which states that $\mathbb{C}[\hat{Y}]$ is multiplicity free. \square

Theorem 15: *Let G be the simply connected simple algebraic group of type E_6 and let $H \subset G$ be the maximal reductive subgroup of type $A_5 \times A_1$.*

Then we have the following branching rules:

$$\begin{aligned} i) \quad \text{res}_H^G(V_{k\omega_1}) &= \bigoplus_{a_1+2a_2+a_3=k} V_{a_1\lambda_2+a_2\lambda_4+a_3\lambda_5} \otimes V_{a_3\lambda_6}, \\ ii) \quad \text{res}_H^G(V_{k\omega_6}) &= \bigoplus_{a_1+2a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_2+a_3\lambda_4} \otimes V_{a_1\lambda_6}. \end{aligned}$$

Remark. In E_6 we have $\omega_1^* = \omega_6$, $\omega_2^* = \omega_2$, $\omega_3^* = \omega_5$ and $\omega_4^* = \omega_4$.

Proof: ii) With “LiE” we compute

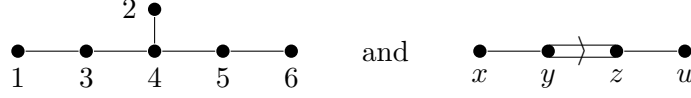
$$\begin{aligned} \text{res}_H^G(V_{\omega_6}) &= (V_{\lambda_4} \otimes \mathbb{C}) \oplus (V_{\lambda_1} \otimes V_{\lambda_6}), \\ \text{res}_H^G(V_{2\omega_6}) &= (V_{2\lambda_4} \otimes \mathbb{C}) \oplus (V_{\lambda_1+\lambda_4} \otimes V_{\lambda_6}) \oplus (V_{2\lambda_1} \otimes V_{2\lambda_6}) \oplus (V_{\lambda_2} \otimes \mathbb{C}). \end{aligned}$$

There are at least two generators of degree 1 and of weights $(\lambda_4, 0)$ and (λ_1, λ_6) and one generator of degree 2 and of weight $(\lambda_2, 0)$ for $\mathbb{C}[\hat{Y}]^{U_H}$ with $Y = G/P_1$. In the proof of the previous theorem we have found an element $X \in N$ with a U_L -orbit of codimension 2. So it follows that $\dim \mathbb{C}[\hat{Y}]^{U_H} = 3$ and the branching rules follow immediately.

i) These branching rules follow directly from ii) by noting that $\omega_1 = \omega_6^*$, $\lambda_1^* = \lambda_5$, $\lambda_2^* = \lambda_4$ and $\lambda_6^* = \lambda_6$. \square

Theorem 16: *Let G be the simply connected simple algebraic group of type E_6 and let H be the maximal reductive subgroup of type F_4 . Then G/P_i , $i \neq 4$, are spherical H -varieties. The variety G/P_4 is not H -spherical.*

Proof: If we have the Dynkin diagrams



of E_6 and F_4 , then we have an embedding of the simple Lie-algebra F_4 in E_6 by choosing the following root vectors

$$\begin{aligned} X_x &:= X_{(0,1,0,0,0,0)}, & X_z &:= \frac{1}{\sqrt{2}}(X_{(0,0,1,0,0,0)} + X_{(0,0,0,0,1,0)}) \\ X_y &:= X_{(0,0,0,1,0,0)}, & X_u &:= \frac{1}{\sqrt{2}}(X_{(1,0,0,0,0,0)} + X_{(0,0,0,0,0,1)}) \end{aligned}$$

([Dyn57, p. 258, Table 24] with different numbering of the Dynkin diagrams). Now we consider the associated algebraic subgroup of E_6 .

Case G/P_1 : We compute

$$N = \mathbb{C}X_{-(1,1,1,2,2,1)}.$$

So N is obviously L -spherical and thus G/P_1 is H -spherical.

Case G/P_6 : The H -sphericity of $Y = G/P_6$ is an immediate corollary of the following theorem which states that $\mathbb{C}[\widehat{Y}]$ is multiplicity free.

Case G/P_2 : In this case we get

$$\begin{aligned} N = & \mathbb{C}X_{-(0,1,0,1,1,0)} \oplus \mathbb{C}X_{-(0,1,0,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,1,1,1)} \oplus \\ & \mathbb{C}X_{-(0,1,1,2,1,1)} \oplus \mathbb{C}X_{-(0,1,1,2,2,1)} \oplus \mathbb{C}X_{-(1,1,1,2,2,1)}_{E_6}. \end{aligned}$$

If we define $X := X_{-(1,1,1,2,2,1)}$ then we have:

$$\begin{aligned} [X_{(0,0,0,1)_{F_4}}, X] &= X_{-(0,1,1,2,2,1)}, & [X_{(0,0,1,1)_{F_4}}, X] &= X_{-(0,1,1,2,1,1)}, \\ [X_{(0,1,1,1)_{F_4}}, X] &= X_{-(0,1,1,1,1,1)}, & [X_{(0,1,2,1)_{F_4}}, X] &= X_{-(0,1,0,1,1,1)}, \\ [X_{(0,1,2,2)_{F_4}}, X] &= X_{-(0,1,0,1,1,0)}. \end{aligned}$$

With $[\mathfrak{h}, X] = \mathbb{C}X$ we get $[\mathfrak{b}, X] = N$ and it follows that N is a spherical L -module.

Case G/P_3 : In this case we get

$$\begin{aligned} N = & \mathbb{C}X_{-(0,0,1,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,2,1,1)} \oplus \\ & \mathbb{C}X_{-(0,1,1,2,2,1)} \oplus \mathbb{C}X_{-(1,1,1,2,2,1)}. \end{aligned}$$

Set $X := X_{-(1,1,1,2,2,1)} + X_{-(0,1,1,2,1,1)}$. Then we have

$$[\mathfrak{h}, X] = \mathbb{C}X_{-(1,1,1,2,2,1)} \oplus \mathbb{C}X_{-(0,1,1,2,1,1)},$$

since the roots of the root vectors defining X are linearly independent. Furthermore we have

$$\begin{aligned} [X_{(0,0,0,1)_{F_4}}, X] &= X_{-(0,1,1,2,2,1)}, & [X_{(1,0,0,0)_{F_4}}, X] &= X_{-(0,1,1,1,1,1)}, \\ [X_{(1,1,0,0)_{F_4}}, X] &= X_{-(0,0,1,1,1,1)}. \end{aligned}$$

So $[\mathfrak{b}, X] = N \Rightarrow N$ is a spherical L -module and this implies that G/P_3 is H -spherical.

Case G/P_5 : The H -sphericity of $Y = G/P_5$ is an immediate corollary of the following theorem which states that $\mathbb{C}[\widehat{Y}]$ is multiplicity free. \square

We can derive branching rules in the cases where G/P_i is a spherical H -variety.

Theorem 17: Let G be the simple simply connected algebraic group of type E_6 and H be the subgroup of type F_4 .

Then we have the branching rules:

$$\begin{aligned}
i) \quad \text{res}_H^G(V_{k\omega_1}) &= \bigoplus_{a_1 \leq k} V_{a_1 \lambda_4}, \\
ii) \quad \text{res}_H^G(V_{k\omega_2}) &= \bigoplus_{a_1 + a_2 = k} V_{a_1 \lambda_1 + a_2 \lambda_4}, \\
iii) \quad \text{res}_H^G(V_{k\omega_3}) &= \bigoplus_{a_1 + a_2 + a_3 = k} V_{a_1 \lambda_1 + a_2 \lambda_3 + a_3 \lambda_4}, \\
iv) \quad \text{res}_H^G(V_{k\omega_5}) &= \bigoplus_{a_1 + a_2 + a_3 = k} V_{a_1 \lambda_1 + a_2 \lambda_3 + a_3 \lambda_4}, \\
v) \quad \text{res}_H^G(V_{k\omega_6}) &= \bigoplus_{a_1 \leq k} V_{a_1 \lambda_4}.
\end{aligned}$$

Proof: v) In this case we work with $Y = G/P_1$. With “LiE” we compute

$$\text{res}_H^G(V_{\omega_6}) = \mathbb{C} \oplus V_{\lambda_4}.$$

Since N is 1-dimensional in this case, each U_L -orbit is 0-dimensional with codimension 1. So $\dim \mathbb{C}[\widehat{Y}]^{U_H} = 2$ and $\mathbb{C}[\widehat{Y}]^{U_H}$ is generated by its degree-1-elements. The branching rules follow.

i) These branching rules follow directly from v) by noting that $\omega_1 = \omega_6^*$ and $\overline{\lambda}_i^* = \lambda_i$.

ii) In this case we work with $Y = G/P_2$. With “LiE” we compute

$$\text{res}_H^G(V_{\omega_2}) = V_{\lambda_1} \oplus V_{\lambda_4},$$

so there are two generators of degree 1. The module N is of dimension 6 and we have seen that $X_{-(1,1,1,2,2,1)} \in N$ is an element such that $U_L.X$ is of dimension 5. So $\dim \mathbb{C}[\widehat{Y}]^{U_H} \leq 2$ and hence $\mathbb{C}[\widehat{Y}]^{U_H}$ is generated by its degree-1-elements. The branching rules follow immediately.

iv) In this case we work with $Y = G/P_3$. With “LiE” we compute

$$\text{res}_H^G(V_{\omega_5}) = V_{\lambda_1} \oplus V_{\lambda_3} \oplus V_{\lambda_4},$$

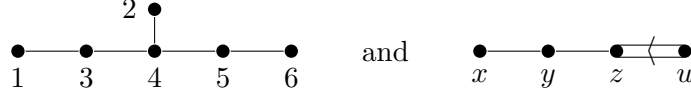
so again there are 3 generators of degree 1. The module N is of dimension 5 and $X_{-(1,1,1,2,2,1)} + X_{-(0,1,1,2,1,1)} \in N$ is an element of N with a 3-dimensional U_L -orbit (cf. proof of previous theorem). So $\dim \mathbb{C}[\widehat{Y}]^{U_H} \leq 3$. It follows that $\mathbb{C}[\widehat{Y}]^{U_H}$ is generated by its degree-1-elements and so the branching rules follow.

iii) These branching rules follow directly from v) by noting that $\omega_3 = \omega_5^*$ and $\overline{\lambda}_i^* = \lambda_i$. \square

Theorem 18: Let G be the simply connected simple algebraic group of type E_6 and let H be the maximal reductive subgroup of type C_4 . Then G/P_1 and G/P_6 are spherical H -varieties. The varieties $G/P_2, \dots, G/P_5$ are not H -spherical.

Proof: That $G/P_2, \dots, G/P_5$ are not H -spherical follows by dimension reasons.

For the other two cases we consider the Dynkin diagrams



of E_6 and C_4 respectively. Then the simple Lie-algebra of type C_4 is embedded into the simple Lie-algebra of type E_6 by choosing the following root vectors:

$$X_x := \frac{1}{\sqrt{2}}(X_{(0,1,1,1,0,0)} + X_{(0,1,0,1,1,0)}), \quad X_y := \frac{1}{\sqrt{2}}(X_{(1,0,0,0,0,0)} + X_{(0,0,0,0,0,1)})$$

$$X_z := \frac{1}{\sqrt{2}}(X_{(0,0,1,0,0,0)} + X_{-(0,0,0,0,1,0)}), \quad X_u := X_{(0,0,0,1,0,0)}$$

(cf. [Dyn57, p. 258, Table 24]). Now we consider the associated subgroup H of G .

Case G/P_1 : We compute

$$N = \mathbb{C}X_{-(1,1,1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,1,2,1,1)} \oplus \mathbb{C}X_{-(1,1,2,2,1,1)} \oplus \mathbb{C}X_{-(1,1,2,2,2,1)} \oplus \mathbb{C}X_{-(1,1,2,3,2,1)}.$$

We define $X := X_{-(1,1,2,3,2,1)} + X_{-(1,1,1,1,1,1)}$. Then we have

$$[\mathfrak{h}, X] = \mathbb{C}X_{-(1,1,2,3,2,1)} \oplus \mathbb{C}X_{-(1,1,1,1,1,1)}.$$

Further we get

$$[X_{(0,0,0,1)_{C_4}}, X] = X_{-(1,1,2,2,2,1)}, \quad [X_{(0,0,1,1)_{C_4}}, X] = X_{-(1,1,2,2,1,1)}$$

$$[X_{(0,0,2,1)_{C_4}}, X] = X_{-(1,1,1,2,1,1)}.$$

This implies that $[\mathfrak{h}, X]$ contains five linearly independent vectors of $N \Rightarrow [\mathfrak{h}, X] = N$. Hence N is L -spherical.

Case G/P_6 : The H -sphericity of $Y = G/P_6$ is an immediate corollary of the following theorem which states that $\mathbb{C}[\hat{Y}]$ is multiplicity free. \square

From the spherical cases we can derive the following branching rules:

Theorem 19: *Let G be the simply connected simple algebraic group of type E_6 and H be the subgroup of type C_4 .*

Then we have the following branching rules:

$$i) \quad \text{res}_H^G(V_{k\omega_1}) = \bigoplus_{a_1+2a_2+2a_3=k} V_{a_1\lambda_2+a_2\lambda_4},$$

$$ii) \quad \text{res}_H^G(V_{k\omega_6}) = \bigoplus_{a_1+2a_2+2a_3=k} V_{a_1\lambda_2+a_2\lambda_4}.$$

Proof:

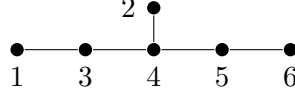
ii) Here we are in the case $Y = G/P_1$. With “LiE” we compute

$$\text{res}_H^G(V_{\omega_6}) = V_{\lambda_2} \quad \text{and} \quad \text{res}_H^G(V_{2\omega_6}) = \mathbb{C} \oplus V_{2\lambda_2} \oplus V_{\lambda_4}.$$

So there is one generator of degree 1 and two of degree 2 in $\mathbb{C}[\hat{Y}]^{U_H}$. From the calculations in the proof of the previous theorem we know that $X_{-(1,1,2,3,2,1)} + X_{-(1,1,1,1,1,1)}$ is an element of N whose U_L -orbit is of codimension 2. Hence $\dim \mathbb{C}[\hat{Y}]^{U_H} \leq 3$. But since we have already found three generators we know that $\dim \mathbb{C}[\hat{Y}]^{U_H} = 3$. The branching rules follow immediately.

i) These branching rules follow directly from ii) by noting that $\omega_1 = \omega_6^*$, $\lambda_2^* = \lambda_2$ and $\lambda_4^* = \lambda_4$. \square

Next we will consider the Levi subgroup H of G that is obtained by omitting the simple root α_1 . From the Dynkin diagram of E_6 we see that H is the group $D_5 \times \mathbb{C}^*$.



Theorem 20: Let G be the simply connected simple algebraic group of type E_6 and let H be the Levi subgroup $D_5 \times \mathbb{C}^*$. Then G/P_i is a spherical H -variety for $i \neq 4$. The variety G/P_4 is not H -spherical.

Proof: This is proven in [Lit94]. \square

Theorem 21: Let G be the simply connected simple algebraic groups of type E_6 let $H \subset G$ be the Levi subgroup $D_5 \times \mathbb{C}^*$. Then we have the following branching rules.

$$\begin{aligned}
i) \quad \text{res}_H^G(V_{k\omega_1}) &= \bigoplus_{a_1+a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_4} \otimes V_{-2a_1+a_2+4a_3}, \\
ii) \quad \text{res}_H^G(V_{k\omega_2}) &= \bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\lambda_2+a_2\lambda_4+a_3\lambda_5} \otimes V_{-3a_2+3a_3}, \\
iii) \quad \text{res}_H^G(V_{k\omega_3}) &= \bigoplus_{a_1+\dots+a_6=k} \frac{V_{(a_1+a_6)\lambda_1+a_2\lambda_2+a_3\lambda_3+(a_4+a_6)\lambda_4+a_5\lambda_5}}{V_{2a_1-4a_2+2a_3+5a_4-a_5-3a_6}}, \\
iv) \quad \text{res}_H^G(V_{k\omega_5}) &= \bigoplus_{a_1+\dots+a_6=k} \frac{V_{(a_1+a_6)\lambda_1+a_2\lambda_2+a_3\lambda_3+a_4\lambda_4+(a_5+a_6)\lambda_5}}{V_{-2a_1+4a_2-2a_3+a_4-5a_5+a_6}}, \\
v) \quad \text{res}_H^G(V_{k\omega_6}) &= \bigoplus_{a_1+a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_5} \otimes V_{2a_1-a_2-4a_3}.
\end{aligned}$$

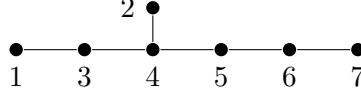
Proof: From paragraph 1.4 in [Lit94] we get the following branching rules.

$$\begin{aligned}
i) \quad \text{res}_H^G(V_{k\omega_1}) &= \bigoplus_{a_1+a_2+a_3=k} V_{(a_3-a_1-a_2)\omega_1+a_2\omega_3+a_1\omega_6}, \\
ii) \quad \text{res}_H^G(V_{k\omega_2}) &= \bigoplus_{a_1+a_2+a_3+a_4=k} V_{-(a_1+2a_2)\omega_1+a_3\omega_2+a_2\omega_3+a_1\omega_5}, \\
iii) \quad \text{res}_H^G(V_{k\omega_3}) &= \bigoplus_{a_1+\dots+a_6=k} \frac{V_{-(2a_2+a_3+a_5+2a_6)\omega_1+a_5\omega_2+(a_4+a_6)\omega_3}}{+a_3\omega_4+a_2\omega_5+(a_1+a_6)\omega_6}, \\
iv) \quad \text{res}_H^G(V_{k\omega_5}) &= \bigoplus_{a_1+\dots+a_6=k} \frac{V_{-(a_1+2a_3+a_4+2a_5+a_6)\omega_1+(a_5+a_6)\omega_2+a_4\omega_3}}{+a_3\omega_4+a_2\omega_5+(a_1+a_6)\omega_6}, \\
v) \quad \text{res}_H^G(V_{k\omega_6}) &= \bigoplus_{a_1+a_2+a_3=k} V_{-(a_2+a_3)\omega_1+a_2\omega_2+a_1\omega_6}.
\end{aligned}$$

We would like to write these highest weights in terms of the fundamental weights of D_5 and \mathbb{C}^* . We have $\omega_6 = \lambda_1$, $\omega_5 = \lambda_2$, $\omega_4 = \lambda_3$, $\omega_3 = \lambda_4$ and $\omega_2 = \lambda_5$, where λ_i are the fundamental weights of D_5 and we fix the coweight $3\omega_1^\vee = 4\alpha_1^\vee + 3\alpha_2^\vee + 5\alpha_3^\vee + 6\alpha_4^\vee + 4\alpha_5^\vee + 2\alpha_6^\vee$ which determines the highest weights for \mathbb{C}^* . Thus we get the branching rules in the theorem. \square

8. THE EXCEPTIONAL GROUP OF TYPE E_7

Let G be of type E_7 with the following Dynkin-Diagram.



For this group there are only a few cases of sphericity as we will see. As we did in the last section we start by calculating the dimensions of the Borel subgroups of the maximal reductive subgroups as well as the dimensions of G/P_i for $i = 1, \dots, 7$.

We have

	G/P_1	G/P_2	G/P_3	G/P_4	G/P_5	G/P_6	G/P_7
dim	33	42	47	53	50	42	27

For the Borel subgroups B_H we have:

H	A_7	$E_6 \times \mathbb{C}^*$	$A_3 \times A_3 \times A_1$	$A_5 \times A_2$	$D_6 \times A_1$	$A_1 \times A_1$
dim B_H	35	43	20	25	38	4

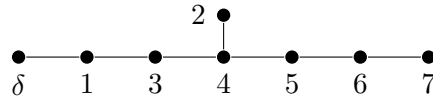
H	$A_1 \times G_2$	$G_2 \times C_3$	$A_1 \times F_4$	A_1	A_2
dim B_H	10	20	30	2	5

So we can rule out a lot of cases by dimension comparison.

Proposition 22: *Let G be the simply connected simple algebraic group of type E_7 . If H is a maximal reductive subgroup of type $A_3 \times A_3 \times A_1$, $A_5 \times A_2$, $A_1 \times A_1$, $A_1 \times G_2$, $G_2 \times C_3$, A_1 or A_2 , then G/P_i is not a spherical H -variety for $i = 1, \dots, 7$. \square*

Proof: In these cases we have $\dim B_H < \dim G/P_i$ for $i = 1, \dots, 7$. \square

Now we turn to the remaining subgroups and start with the subgroup of type A_7 . This is a subsystem subgroup so we add the smallest root δ to the simple roots and consider the extended Dynkin diagram.



By omitting the simple root α_2 we obtain the embedding of the root system A_7 into E_7 . Explicitly we get

$$\begin{aligned}
 (1, 0, 0, 0, 0, 0, 0)_{A_7} &= (1, 0, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0)_{A_7} = (0, 0, 1, 0, 0, 0, 0), \\
 (0, 0, 1, 0, 0, 0, 0)_{A_7} &= (0, 0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0, 0)_{A_7} = (0, 0, 0, 0, 1, 0, 0), \\
 (0, 0, 0, 0, 1, 0, 0)_{A_7} &= (0, 0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 1, 0)_{A_7} = (0, 0, 0, 0, 0, 0, 1), \\
 (0, 0, 0, 0, 0, 0, 1)_{A_7} &= (1, 2, 2, 3, 2, 1, 0).
 \end{aligned}$$

Now we consider the corresponding subsystem subgroup H .

Theorem 23: *Let G be the simply connected simple algebraic group of type E_7 and H the maximal reductive subgroup of type A_7 . Then G/P_7 is a spherical H -variety whereas G/P_i is not H -spherical for $i \neq 7$.*

Proof: By dimension comparison G/P_i can only be spherical for $i = 1$ or $i = 7$. We know that for E_7 we have $\omega_i^* = \omega_i$. And with LiE we compute

$$\text{res}_H^G(V_{4\omega_1}) = \dots \oplus 2V_{\lambda_4} \oplus \dots$$

This shows that we have multiplicities in this case and G/P_1 is not a spherical H -variety.

For G/P_7 we use the same methods as above. We compute

$$\begin{aligned} N = & \mathbb{C}X_{-(0,1,0,1,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,1,1,1,1,1)} \oplus \\ & \mathbb{C}X_{-(0,1,1,2,1,1,1)} \oplus \mathbb{C}X_{-(1,1,1,2,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,2,2,1,1)} \oplus \\ & \mathbb{C}X_{-(1,1,2,2,1,1,1)} \oplus \mathbb{C}X_{-(1,1,1,2,2,1,1)} \oplus \mathbb{C}X_{-(0,1,1,2,2,2,1)} \oplus \\ & \mathbb{C}X_{-(1,1,2,2,2,1,1)} \oplus \mathbb{C}X_{-(1,1,1,2,2,2,1)} \oplus \mathbb{C}X_{-(1,1,2,3,2,1,1)} \oplus \\ & \mathbb{C}X_{-(1,1,2,2,2,2,1)} \oplus \mathbb{C}X_{-(1,1,2,3,2,2,1)} \oplus \mathbb{C}X_{-(1,1,2,3,3,2,1)}. \end{aligned}$$

Define $X := X_{-(1,1,2,3,3,2,1)} + X_{-(1,1,2,2,1,1,1)} + X_{-(0,1,0,1,1,1,1)}$. The roots of the root-vectors in X are linearly independent. Thus we get that

$$[\mathfrak{h}, X] := \langle X_{-(1,1,2,3,3,2,1)_{E_7}}, X_{-(1,1,2,2,1,1,1)_{E_7}}, X_{-(0,1,0,1,1,1,1)_{E_7}} \rangle$$

and further

$$\begin{aligned} [X_{(0,0,1,0,0,0,0)}, X] &= X_{-(1,1,1,2,1,1,1)}, & [X_{(0,0,0,0,1,0,0)}, X] &= X_{-(1,1,2,3,2,2,1)}, \\ [X_{(1,0,1,0,0,0,0)}, X] &= X_{-(0,1,1,2,1,1,1)}, & [X_{(0,0,1,1,0,0,0)}, X] &= X_{-(1,1,1,1,1,1,1)}, \\ [X_{(0,0,0,1,1,0,0)}, X] &= X_{-(1,1,2,2,2,2,1)}, & [X_{(0,0,0,0,1,1,0)}, X] &= X_{-(1,1,2,3,2,1,1)}, \\ [X_{(1,0,1,1,0,0,0)}, X] &= X_{-(0,1,1,1,1,1,1)}, & [X_{(0,0,1,1,1,0,0)}, X] &= X_{-(1,1,1,2,2,2,1)}, \\ [X_{(0,0,0,1,1,1,0)}, X] &= X_{-(1,1,2,2,2,1,1)}, & [X_{(1,0,1,1,1,0,0)}, X] &= X_{-(0,1,1,2,2,2,1)}, \\ [X_{(0,0,1,1,1,1,0)}, X] &= X_{-(1,1,1,2,2,1,1)}, & [X_{(1,0,1,1,1,1,0)}, X] &= X_{-(0,1,1,2,2,1,1)}. \end{aligned}$$

This shows that $\dim[\mathfrak{h}, X] = 15 = \dim N \Rightarrow [\mathfrak{h}, X] = N \Rightarrow N$ is a spherical L -module. And thus G/P_7 is a spherical H -variety. \square

Since G/P_7 is a spherical H -variety we can derive branching rules for $V_{k\omega_7^*} = V_{k\omega_7}$.

Theorem 24: *Let G be the simply connected simple algebraic group of type E_7 and H the maximal reductive subgroup of type A_7 . Then*

$$\text{res}_H^G(V_{k\omega_7}) = \bigoplus_{2a_1+a_2+2a_3+a_4=k} V_{a_2\lambda_2+a_3\lambda_4+a_4\lambda_6}.$$

Proof: With “LiE” we compute

$$\text{res}_H^G(V_{\omega_7}) = V_{\lambda_2} \oplus V_{\lambda_6}.$$

So there are two generators of degree 1 of weight λ_2 and λ_6 . Further we have

$$\text{res}_H^G(V_{2\omega_7}) = \mathbb{C} \oplus V_{2\lambda_2} \oplus V_{2\lambda_6} \oplus V_{\lambda_2+\lambda_6} \oplus V_{\lambda_4},$$

which shows that there are 2 generators of degree 2 which are of weight 0 and λ_4 . This shows that $\dim \mathbb{C}[\hat{Y}]^{U_H} \geq 4$.

In the proof of the previous theorem we have found an $X \in N$ such that $U_L.X$ is of codimension 3. It follows that $\dim \mathbb{C}[\hat{Y}]^{U_H} = 4$ and we have found four generators. The branching rules follow immediately. \square

Next we will consider the Levi subgroup $E_6 \times \mathbb{C}^*$, which is obtained by omitting the simple root α_7 in the Dynkin-diagram.

Theorem 25: *Let G be the simply connected simple algebraic group of type E_7 and $H \subset G$ the Levi subgroup of type $E_6 \times \mathbb{C}^*$. Then G/P_1 and G/P_7 are spherical H -varieties whereas G/P_i , $i = 2, \dots, 6$ are not spherical H -varieties.*

Proof: This was proven in [Lit94]. \square

We get the following branching rules from the spherical cases.

Theorem 26: *Let G be the simply connected simple algebraic group of type E_7 and H the Levi subgroup of type $E_6 \times \mathbb{C}^*$. Then we have the following branching rules.*

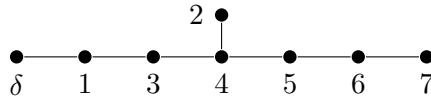
$$\begin{aligned} \text{i)} \quad \text{res}_H^G(V_{k\omega_1}) &= \bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\lambda_1+a_2\lambda_2+a_3\lambda_6} \otimes V_{2a_1-2a_3}, \\ \text{ii)} \quad \text{res}_H^G(V_{k\omega_2}) &= \bigoplus_{\substack{a_1+a_2+a_3+2a_4+ \\ a_5+a_6+a_7=k}} V_{a_1\lambda_1+(a_2+a_7)\lambda_2+a_3\lambda_3+a_4\lambda_4+a_5\lambda_5+a_6\lambda_6} \otimes V_{-a_1+3a_2+a_3-a_5-2a_6}, \\ \text{iii)} \quad \text{res}_H^G(V_{k\omega_7}) &= \bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\lambda_1+a_2\lambda_6} \otimes V_{-a_1+a_2+3a_3-3a_4}. \end{aligned}$$

Proof: From paragraph 1.4 in [Lit94] we get the following branching rules.

$$\begin{aligned} \text{i)} \quad \text{res}_H^G(V_{k\omega_1}) &= \bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\omega_1+a_2\omega_2+a_3\omega_6-(a_2+2a_3)\omega_7}, \\ \text{ii)} \quad \text{res}_H^G(V_{k\omega_2}) &= \bigoplus_{\substack{a_1+a_2+a_3+2a_4+ \\ a_5+a_6+a_7=k}} V_{a_1\omega_1+(a_2+a_7)\omega_2+a_3\omega_3+a_4\omega_4+a_5\omega_5+a_6\omega_6 - (a_1+a_3+2a_4+2a_5+2a_6+a_7)\omega_7}, \\ \text{iii)} \quad \text{res}_H^G(V_{k\omega_7}) &= \bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\omega_1+a_2\omega_6+(a_3-a_1-a_2-a_4)\omega_7}. \end{aligned}$$

We have $\omega_i = \lambda_i$ for $i = 1, \dots, 6$ and we fix the coweight $2\omega_7^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee + 4\alpha_3^\vee + 6\alpha_4^\vee + 5\alpha_5^\vee + 4\alpha_6^\vee + 3\alpha_7^\vee$ which determines the highest weights for \mathbb{C}^* . Thus we get the branching rules in the theorem. \square

Now we will turn to the subgroup of type $D_6 \times A_1$. We will consider the extended Dynkin-diagram of E_7 again by adding the smallest root δ to the simple roots.



If we omit the simple root α_6 we have a sub-diagram of type $D_6 \times A_1$ and consider the corresponding subsystem subgroup. Explicitly we can choose the following simple roots:

$$\begin{aligned} (1, 0, 0, 0, 0, 0)_H &= (0, 1, 1, 2, 2, 2, 1), \quad (0, 1, 0, 0, 0, 0, 0)_H = (1, 0, 0, 0, 0, 0, 0), \\ (0, 0, 1, 0, 0, 0, 0)_H &= (0, 0, 1, 0, 0, 0, 0), \quad (0, 0, 0, 1, 0, 0, 0)_H = (0, 0, 0, 1, 0, 0, 0), \\ (0, 0, 0, 0, 1, 0, 0)_H &= (0, 1, 0, 0, 0, 0, 0), \quad (0, 0, 0, 0, 0, 1, 0)_H = (0, 0, 0, 0, 1, 0, 0), \\ (0, 0, 0, 0, 0, 0, 1)_H &= (0, 0, 0, 0, 0, 0, 1). \end{aligned}$$

Theorem 27: Let G be the simply connected simple algebraic group of type E_7 . If H is the subgroup of type $D_6 \times A_1$ then G/P_7 is a spherical H -variety and G/P_i is not a spherical H -variety for $i = 1, \dots, 6$.

Proof: Dimension comparison shows that $G/P_2, \dots, G/P_6$ are not H -spherical. For G/P_1 we can compute the restriction of $V_{k\omega_1}$ (note that $\omega_i^* = \omega_i$ for E_7) with LiE and get

$$\text{res}_H^G(V_{4\omega_1}) = \dots \oplus 2(V_{2\lambda_6} \otimes V_{2\lambda_7}) \oplus \dots$$

Thus there are multiplicities in this case and we know that the H -variety G/P_1 is not H -spherical.

Case G/P_7 : We compute

$$\begin{aligned} N = & \mathbb{C}X_{-(0,0,0,0,0,1,1)} \oplus \mathbb{C}X_{-(0,0,0,0,1,1,1)} \oplus \mathbb{C}X_{-(0,0,0,1,1,1,1)} \oplus \\ & \mathbb{C}X_{-(0,1,0,1,1,1,1)} \oplus \mathbb{C}X_{-(0,0,1,1,1,1,1)} \oplus \mathbb{C}X_{-(1,0,1,1,1,1,1)} \oplus \\ & \mathbb{C}X_{-(0,1,1,1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,1,1,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,2,1,1,1)} \oplus \\ & \mathbb{C}X_{-(1,1,1,2,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,2,2,1,1)} \oplus \mathbb{C}X_{-(1,1,2,2,1,1,1)} \oplus \\ & \mathbb{C}X_{-(1,1,1,2,2,1,1)} \oplus \mathbb{C}X_{-(1,1,2,2,2,1,1)} \oplus \mathbb{C}X_{-(1,1,2,3,2,1,1)} \oplus \\ & \mathbb{C}X_{-(1,2,2,3,2,1,1)}. \end{aligned}$$

Now define $X := X_{-(1,2,2,3,2,1,1)} + X_{-(1,0,1,1,1,1,1)}$. The roots of these two root vectors are linearly independent and we have

$$[\mathfrak{h}, X] = \langle X_{-(1,2,2,3,2,1,1)}, X_{-(1,0,1,1,1,1,1)} \rangle$$

Further we have

$$\begin{aligned} [X_{(1,0,0,0,0,0,0)}, X] &= X_{-(0,0,1,1,1,1,1)}, [X_{(0,1,0,0,0,0,0)}, X] = X_{-(1,1,2,3,2,1,1)}, \\ [X_{(1,0,1,0,0,0,0)}, X] &= X_{-(0,0,0,1,1,1,1)}, [X_{(0,1,0,1,0,0,0)}, X] = X_{-(1,1,2,2,2,1,1)}, \\ [X_{(1,0,1,1,0,0,0)}, X] &= X_{-(0,0,0,0,1,1,1)}, [X_{(0,1,1,1,0,0,0)}, X] = X_{-(1,1,1,2,2,1,1)}, \\ [X_{(0,1,0,1,1,0,0)}, X] &= X_{-(1,1,2,2,1,1,1)}, [X_{(1,1,1,1,0,0,0)}, X] = X_{-(0,1,1,2,2,1,1)}, \\ [X_{(1,0,1,1,1,0,0)}, X] &= X_{-(0,0,0,0,0,1,1)}, [X_{(0,1,1,1,1,0,0)}, X] = X_{-(1,1,1,2,1,1,1)}, \\ [X_{(1,1,1,1,1,0,0)}, X] &= X_{-(0,1,1,2,1,1,1)}, [X_{(0,1,1,2,1,0,0)}, X] = X_{-(1,1,1,1,1,1,1)}, \\ [X_{(1,1,1,2,1,0,0)}, X] &= X_{-(0,1,1,1,1,1,1)}, [X_{(1,1,2,2,1,0,0)}, X] = X_{-(0,1,0,1,1,1,1)}. \end{aligned}$$

So we have $\dim[\mathfrak{b}, X] = 16 = \dim N$. This implies that N is a spherical L -module and thus G/P_7 is a spherical H -variety. \square

From the sphericity of G/P_7 we can derive branching rules for $V_{k\omega_7^*} = V_{k\omega_7}$.

Theorem 28: Let G be the simply connected simple algebraic group of type E_7 and let H be a maximal reductive subgroup of type $D_6 \times A_1$. Then

$$\text{res}_H^G(V_{k\omega_7}) = \bigoplus_{a_1+2a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_2+a_3\lambda_6} \otimes V_{a_1\lambda_7}.$$

Proof: With “LiE” we compute

$$\text{res}_H^G(V_{\omega_7}) = (V_{\lambda_1} \otimes V_{\lambda_7}) \oplus (V_{\lambda_6} \otimes \mathbb{C}).$$

So there are two generators of degree 1 with weights (λ_1, λ_7) and $(\lambda_6, 0)$. Further we have

$$\text{res}_H^G(V_{2\omega_7}) = (V_{2\lambda_1} \otimes V_{2\lambda_7}) \oplus (V_{\lambda_1+\lambda_6} \otimes V_{\lambda_7}) \oplus (V_{2\lambda_6} \otimes \mathbb{C}) \oplus (V_{\lambda_2} \otimes \mathbb{C}).$$

Thus there is a further generator of degree 2 and weight λ_2 and we know that $\dim \mathbb{C}[\widehat{Y}]^{U_H} \geq 3$.

In the proof of the previous theorem we have seen that there is an $X \in N$ such that $\dim U_H.X$ is of codimension 2. It follows that $\dim \mathbb{C}[\widehat{Y}]^{U_H} = 3$. The branching rules follow. \square

The last maximal reductive subgroup of G where a sphericity of G/P_i can occur is the group H of type $A_1 \times F_4$. From the table with dimensions of G/P_i we know that only G/P_7 can be a spherical H -variety. But with LiE we compute

$$\text{res}_H^G(V_{4\omega_7}) = \dots \oplus 2(V_{4\lambda_1} \otimes V_{\lambda_5}) \oplus \dots$$

and thus there are multiplicities in this case. We have shown:

Theorem 29: *Let G be the simply connected simple group of type E_7 and H the maximal subgroup of type $A_1 \times F_4$.*

Then G/P_i ($i = 1, \dots, 7$) is not a spherical variety. \square

9. THE EXCEPTIONAL GROUP OF TYPE E_8

We start our computations again by calculating the dimensions of the Borel subgroups of the maximal reductive subgroups and the dimensions of G/P_i for $i = 1, \dots, 8$.

H	$E_7 \times A_1$	$E_6 \times A_2$	$A_3 \times D_5$	$A_4 \times A_4$	$A_5 \times A_2 \times A_1$
$\dim B_H$	72	47	34	28	27

H	$A_7 \times A_1$	D_8	A_8	$G_2 \times F_4$	$A_2 \times A_1$	C_2	A_1
$\dim B_H$	37	72	44	36	6	6	2

The dimensions of the varieties G/P_i ($i = 1, \dots, 8$) are:

	G/P_1	G/P_2	G/P_3	G/P_4	G/P_5	G/P_6	G/P_7	G/P_8
\dim	78	92	98	106	104	97	83	57

By dimension comparison there are only two possibilities of sphericity. If we take the maximal reductive subgroup H_1 of type $E_7 \times A_1$ or the maximal reductive subgroup H_2 of type D_8 , then the variety G/P_8 can be spherical for H_1 or H_2 . But we can compute the following restrictions by using LiE

$$\text{res}_{H_1}^G(V_{5\omega_8}) = \dots \oplus 2(V_{\lambda_1+2\lambda_7} \otimes V_{2\lambda_8}) \oplus \dots,$$

$$\text{res}_{H_2}^G(V_{4\omega_8}) = \dots \oplus 2V_{\lambda_8} \oplus \dots,$$

which show that there are multiplicities in these cases. So there are no spherical cases for G . We have shown:

Theorem 30: *Let G be the simply connected simple algebraic groups of type E_8 . Let H be one of its maximal reductive subgroups.*

Then G/P_i ($i = 1, \dots, 8$) is not a spherical variety. \square

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